GEOMETRICAL MODES: A NEW CONCEPT OF MUSICAL COLOR

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Abstract. Taking into account epistemological uses [8, 7] and formal definitions of musical modes, some new properties will arise from their geometrical representation and their parsimonious structure of transformations of what the author calls the "brightness coefficient".

These properties articulate the analogous concept of what, in color theory, the complementary color is. Some properties of geometrical orientation in modes of scales that are maximally and non-maximally even will also be shown, and, finally, a possible implication for what constitutes a "discrete harmonic quality".

Keywords: Music Theory, Mathematical Music Theory, Geometric Music Theory, Modal Theory, Musical Modes, Jazz Harmony, Music Analysis, Music Cognition, Music Perception, Scale Theory, Maximally Even Scales, Geometrical Modes, Brightness Coefficient, Contemporary Music

1. introduction

The concept of musical mode has been evolving since antiquity. From its use in different cultures such as Chinese, Indian, African, Arab, and Greek as a germinal object to form "simple" melodies that can transmit a message, represent emotions, hours of the day, different seasons [8], and even metaphysical and spiritual states [7, 4], to its modern use in romanticism, impresionism, modernism, post-modernism, etc., where the concept of harmony expanded to some other perspectives, such as what it is known as neo-Riemannian theory [10, 2], formalized by David Lewin [5], George Russel's jazz theory presented in his book "*Lydian Chromatic Concept of Tonal Organization*" [12] and different approaches of how harmonic structures behave, and what their properties are [3, 6].

From these perspectives, the concept of harmony has been transforming to be thought about in terms of "*color*", "*sensation*", "*atmosphere*", "*quality*", etc., to name some of them that clearly surpass the domain of what music seems to be objectively (as a deterministic quality). These terms, which describe complex concepts in their abstract form and are subjectively categorized by the perception of the affection they produce, were called "*cross-domain mapping*" by Zbikowski [16]. Interesting questions about the use of these terms are: ¿Which are the things that produce these affections, and why? ¿Is it just a cultural aspect, or does it have some natural qualities?

We can see some examples of harmonic concepts used in contemporary classical and non-classical music, such as John Luther-Adams, Bela Bartók, Michael

Van Der Aa, Chick Corea, to mention some of them, that are non-functional, not completely dodecaphonic, non-serialist, and even non-catalogued as structures that can be named with conventional nomenclature (see Section 3). Regardless of this, a quality in harmonic structures can be perceived, like some character that is possibly formed by the intervalic conformation. This quality, which is not fully described by different theoretical tools such as harmonic nomenclature, intervalic vectors, functionality, voice leading, transformational theory, pitch set, etc., can lead us to think that intervalic descriptions of harmonic structures can form a way of abstraction to form a "harmonic concept": a geometrical, spatial-related conformation.

An interesting way to explore these aspects arises from geometry and from some properties of their intervallic structural description, their brightness coefficient, and transformational aspects of these two qualities. It is important to take into account that harmonic structure is taken as an object that can be built over time, like a conjunction of different small elements, which are pitches; every harmonic structure does not have to be presented in its totality at every moment. The construction of any harmonic structure can (and, for musical effects, is preferable) display the structure in a span of time.

The conjunction of different harmonic entities (notes, chords, and so on) displayed in a span of time can build a structure which we can analogize to what Dimitri Tymoczko calls "*macroharmony*" in his book, *A Geometry of Music*, that is

> "*the total collection of notes heard over moderate spans of musical time. Tonal music tends to use relatively small macroharmonies, often involving five to eight notes.*" [14].

It must not be mistaken for the other significance of *macroharmony* in spectralism [11].

Then a harmonic structure built on a span of time creates a geometrical structure relative to it. This means that the variation of these harmonic entities creates a space where geometrical objects transform into others, which have special properties, indeterminacy states, and qualities based on superposition, which can give us a perspective of what happens in polymodal or polytonal fragments from Stravinsky, Bartók, Charles Ives, to mention some of them, and a possible explanation of what creates quality in a harmonic structure.

2. MAJOR SCALE MODES

To begin the transformational process of what happens in a scale to derive from it a modal structure, it must be clear how a scale is built.

Definition 2.1. *A scale S is defined as an unordered set of $n \leq 12$ elements in modulus-12 that are different from one another:*

$$S := \{s_0 \dots s_{n-1} | \exists! s_n = s_n\}$$

Where $mod n(n-1) \wedge mod 12(s_n) \forall s_n \in S$.

We can have three different categories of scales, which are Chromatic Scale C, General Scale S, and Diatonic Scale D, where the only difference between them is the size n of each one, which is

(1) Chromatic Scale: $|C| = 12$,
(2) General Scale: $|S| \leq 12$,
(3) Diatonic Scale: $|D| = 7$.

Proposition 2.2. *$D \subset C$ and is always size $|D| = 7$, therefore is asymmetric.*

Proof. Let a diatonic scale be a set with cardinality $|D| = 7$ and a dodecaphonic scale another set with cardinality $|C| = 12$, both of which are in modulus-12 for Definition 2.1. The mapping from $D \longrightarrow C$ is always asymmetric (there is more than one kind of interval in its structure) as a direct consequence of their cardinalities' non-divisibility. This applies for any scale S whose interval sequence is non-periodic. □

The implication of the asymmetric structure of the mapping of $D \longrightarrow C$ is to establish a hierarchy. This arises through harmonic functionality, based on the tethical, antithetical, and synthetical functions, respectively [10]. It must be noted that though asymmetry in the scale is a direct consequence of the mapping and also from the acoustical [13] and Pythagorean logic [9] formation of the scale, it is also a necessary state for it to form differentiable harmonic structures, and therefore, establish a hierarchy (in the general form).

The structure of a diatonic scale naturally implies modal structures which arise from the scale. This relationship can expand the conceptual object of a diatonic scale into a geometric one that has different possible hierarchical points with different qualities, some of which we call "colors".

To start introducing the nature of modes in a geometrical context, the structural qualities of four important heptatonic scales (major scale, harmonic minor scale, melodic minor scale, and harmonic major scale) will be analyzed. We chose these scales because they can be found in almost all of the music that is built with 12-tone tuning that is not atonal.

The four heptatonic diatonic scales (major scale D_M, harmonic minor scale D_{mh}, melodic minor scale D_{mm}, and harmonic major scale D_{Mh}) are given as follows:

- $D_M = \{0, 2, 4, 5, 7, 9, 11\}$
- $D_{mh} = \{0, 2, 3, 5, 7, 8, 11\}$
- $D_{mm} = \{0, 2, 3, 5, 7, 9, 11\}$
- $D_{Mh} = \{0, 2, 4, 5, 7, 8, 11\}$

Definition 2.3. *A mode μ is defined as the set of intervals that arise from the cyclical rotation of S. Given a scale $S : \{s_0, s_1 \dots s_{n\leq 12}\}$ a mode is defined as:*

$$\mu_u(S) := \{mod 12(s_n - s_u) | n = \{0, 1 \dots n - 1\}\}$$

where each $u \leq |S|$ is a fixed different mode index.

Lemma 2.4. *The size of a mode $|\mu|$ and the size of the set of different modes $|M|$ within a single scale is equal to the size of the scale $|S|$.*

Proof. Every mode is $\mu(S) \subseteq S$, which always has the same cardinality: $|\mu(S)| = |S|$ for Definition 2.3. Also, there is only one mode for each mode index: $\forall u \exists! \mu_u$; therefore, we have that

$$\forall \mu \subseteq S \wedge \forall u \exists! \mu \implies |\mu| = |M| = |S|$$

□

Some examples of the construction of modes as described in Definition 2.3 will be shown below:

- $\mu_0(S_M) = \{(0-0), (2-0), (4-0), (5-0), (7-0), (9-0), (11-0)\} = \{0, 2, 4, 5, 7, 9, 11\}$
- $\mu_3(S_M) = \{(0-5), (2-5), (4-5), (5-5), (7-5), (9-5), (11-5)\} = \{7, 9, 11, 0, 2, 4, 6\}$
- $\mu_5(S_{mm}) = \{(0-9), (2-9), (3-9), (5-9), (7-9), (9-9), (11-9)\} = \{3, 5, 6, 8, 10, 0, 2\}$

Proposition 2.5. *Every mode of a diatonic scale D is unique in its structure:*

$$\exists! \mu_u(D) = \mu_u(D)$$

Proof. For Definition 2.1, a D scale complies with the condition that 1) every element is different from one another: $\exists! s_n = s_n$, and 2) is asymmetric for Proposition 2.2. Therefore, applying $s_n - s_u$ to every $s_n \in D$ as described in Definition 2.3 will always give a unique structure. □

This way, the modes that arise in each scale are shown in Table 1. It must be noted that every μ is re-ordered from minor to major for clarity purposes.

Scale	Mode index (u)	s_u	$\mu_u(S)$
D_M	0	0	{0, 2, 4, 5, 7, 9, 11}
D_M	1	2	{0, 2, 3, 5, 7, 9, 10}
D_M	2	4	{0, 1, 3, 5, 7, 8, 10}
D_M	3	5	{0, 2, 4, 6, 7, 9, 11}
D_M	4	7	{0, 2, 4, 5, 7, 9, 10}
D_M	5	9	{0, 2, 3, 5, 7, 8, 10}
D_M	6	11	{0, 1, 3, 5, 6, 8, 10}
D_{mh}	0	0	{0, 2, 3, 5, 7, 8, 11}
D_{mh}	1	2	{0, 1, 3, 5, 6, 9, 10}
D_{mh}	2	3	{0, 2, 4, 5, 8, 9, 11}
D_{mh}	3	5	{0, 2, 3, 6, 7, 9, 10}
D_{mh}	4	7	{0, 1, 4, 5, 7, 8, 10}
D_{mh}	5	8	{0, 3, 4, 6, 7, 9, 11}
D_{mh}	6	11	{0, 1, 3, 4, 6, 8, 9}
D_{mm}	0	0	{0, 2, 3, 5, 7, 9, 11}
D_{mm}	1	2	{0, 1, 3, 5, 7, 9, 10}
D_{mm}	2	3	{0, 2, 4, 6, 8, 9, 11}
D_{mm}	3	5	{0, 2, 4, 6, 7, 9, 10}
D_{mm}	4	7	{0, 2, 4, 5, 7, 8, 10}
D_{mm}	5	9	{0, 2, 3, 5, 6, 8, 10}
D_{mm}	6	11	{0, 1, 3, 4, 6, 8, 10}
D_{Mh}	0	0	{0, 2, 4, 5, 7, 8, 11}
D_{Mh}	1	2	{0, 2, 3, 5, 6, 9, 10}
D_{Mh}	2	4	{0, 1, 3, 4, 7, 8, 10}
D_{Mh}	3	5	{0, 2, 3, 6, 7, 9, 11}
D_{Mh}	4	7	{0, 1, 4, 5, 7, 9, 10}
D_{Mh}	5	8	{0, 3, 4, 6, 8, 9, 11}
D_{Mh}	6	11	{0, 1, 3, 5, 6, 8, 9}

TABLE 1. Modes $\mu_u(S)$ of 4 different scales

Once we have defined what a mode is, the next step is to categorize each one of them. This will be done using a theoretical tool, which is harmonic nomenclature.

3. HARMONIC NOMENCLATURE

Modes have different intervallic qualities that differentiate each one of them from the others. We will have seven main modal categories, which arise from the mode of the major scale and are conventionally named as follows:

(1) Ionian ($\mathcal{I}$) $\Leftrightarrow \mu_0(D_M)$,
(2) Dorian ($\mathcal{D}$) $\Leftrightarrow \mu_1(D_M)$,
(3) Phrygian ($\mathcal{P}$) $\Leftrightarrow \mu_2(D_M)$,
(4) Lydian ($\mathcal{L}$) $\Leftrightarrow \mu_3(D_M)$,
(5) Mixolydian ($\mathcal{X}$) $\Leftrightarrow \mu_4(D_M)$,
(6) Aeolian ($\mathcal{A}$) $\Leftrightarrow \mu_5(D_M)$ and
(7) Locrian ($\mathcal{O}$) $\Leftrightarrow \mu_6(D_M)$.

Each one of them is characterized by the intervallic quality arising from the set of intervals of what we will call a "linear chord", which can describe a full structure

and differentiate it from the other ones with a small amount of symbols (see Table 2).

Definition 3.1. *A linear chord* $\overrightarrow{H}$ *is an ordered set formed by the fully extended structure (that covers a linear construction* $\overrightarrow{H} : [\hat{1}, \hat{1}\hat{3}]$*) of notes of stacked thirds (the distance* $n(s_{n+1}) - n(s_n) = 2$*). This chord is built as follows:*

$$\overrightarrow{H} : s_0 \to s_2 \to s_4 \to s_6 \to s_1 \to s_3 \to s_5, \qquad s_n \in \mu$$

Where every $(n)s_n$ *is in modulus-7 for Definition 2.1.*

In the general form of harmonic nomenclature, the indices are not displayed in a modular way; the full extension of a chord in general music theory is then displayed using *degrees*, which have equivalent values:

$$\overrightarrow{H} : \hat{1} \to \hat{3} \to \hat{5} \to \hat{7} \to \hat{9} \to \hat{1}\hat{1} \to \hat{1}\hat{3}$$

where $\hat{h} \in \overrightarrow{H}$ are chord degrees displayed in a non-modular way where $\hat{h} \in \mathbb{N} \setminus 0$. They are all equivalent to their respective s_n, ($s_0 \mapsto \hat{1}$, $s_1 \mapsto \hat{2}$ and so on). This way, when we refer to an $n(s_n)$, it is always mod 12, and on the other hand, $\hat{h} \geq 1$

The construction of the linear chord nomenclature is based on what we call "implications". This is a set of intervals with specific qualities and linear (triadic) formation.

Definition 3.2. *The set of implications that is evaluated in a linear chord* $I(\overrightarrow{H})$ *is the ordered set of intervals in the conventional form* $i = q, s$ *(which can be P1 for perfect 1th, M2 for major second, P5 for perfect fifth, and so on), that are the quality and index of each interval, respectively, where* $i(s_0 - s_0) = P1, i(s_1 - s_0) > P1$*, and so on.*

$I(\overrightarrow{H})$ *is the set of intervals of a chord that is*

- *fully extended: that covers a linear construction* $\overrightarrow{H} : [\hat{1}, \hat{1}\hat{3}]$*,*
- *heptatonic:* $\forall \mu \subseteq D \implies |\mu| = |D| = |M| = 7$*,*
- *diatonic:* $\forall s_n \in \mu \wedge \forall \mu \subseteq D$
- *non-altered in its nomenclature* $\overrightarrow{H} = H13$*, where* H *is the structure that has an* h *basenote.*

The set of intervalic implications is:

$$I(\overrightarrow{H}) : [P1, M3, P5, m7, M9, P11, M13].$$

A non-altered chord $H13$, which is the one that follows each of the implications for each extension of the structure in its nomenclature, could be formed as follows:

- $G13 = [g, b, d, f, a, c, e]$
- $C\sharp 13 = [c\sharp, e\sharp, g\sharp, b, d\sharp, f\sharp, a\sharp]$
- $D\flat\flat 13 = [d\flat\flat, f\flat, a\flat\flat, c\flat\flat, e\flat\flat, g\flat\flat, b\flat\flat]$

Lemma 3.3. *We say an interval* $i^\star$ *is altered when* $q(i^\star) \neq Ps \vee Ms < q(i^\star) < ms$*, where* s *denotes the interval species and* q *its quality. This description will be useful until Section 6.*

The different possible descriptions of a heptatonic linear fully-extended chord will be shown below with an h base note for the general chord (it could be any other pitch). The four parts of the structure are indicated with sub brackets and

the possible symbols within parentheses.

$$\underbrace{(H, H-, H^0, H+)}_{triad}\ \underbrace{(\triangle, ^{\emptyset})}_{7th}\ \underbrace{(9, 11, 13)}_{natural}\ \underbrace{(\flat/\sharp 9, \flat/\sharp 11, \flat/\sharp 13)}_{alterations}$$

Where

- H=Major Triad, H-=Minor Triad, H^0=Diminished Triad, H+=Augmented triad,
- $\triangle$=M7, $^{\emptyset}$=Half-diminished Tetrad
- $9 = \hat{9} \simeq \hat{2}$, $11 = \hat{11} \simeq \hat{4}$, $13 = \hat{13} \simeq \hat{6}$,
- $(\flat/\sharp)$=(-/+) a semitone respect to $I(\overrightarrow{H})$ for each $\hat{h}$.

Every chord that does not follow an implication for any part of the structure must be indicated. In Figure 1, examples of chord nomenclature written in sheet music will be shown.

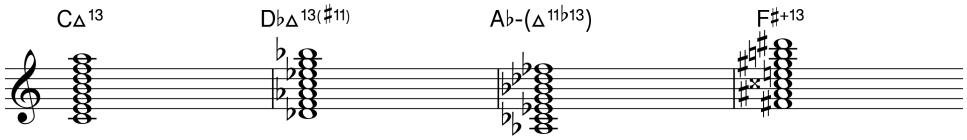


FIGURE 1. Examples of harmonic structure nomenclature

As shown, chords that do not have tones that correspond to the set of implications $I(\overrightarrow{H})$ are indicated either as an additional extension or as a modification of the $\hat{7}$. In Table 2, each mode of the four scales presented will be described using this method.

The nomenclature of modes uses a relatively small set of symbols to indicate the intervallic quality of a mode. This can also give us very useful information about their transformational distance, which can help explain how color notes arise and how modes can be combined to clarify how some mode nomenclature arises from combining two mode names (see Section 4). Geometrical conformation of modes in a superimposed geometrical state can complement this idea (see Section 4).

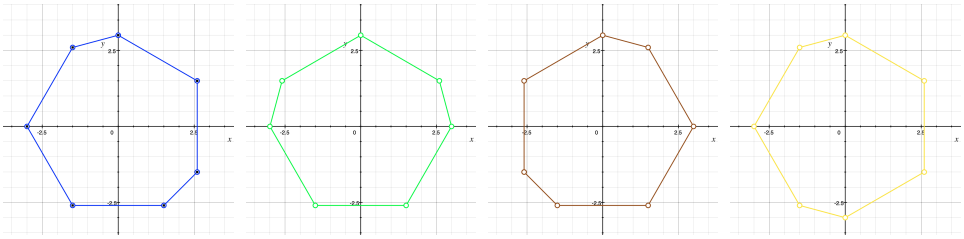
4. GEOMETRICAL MODES AND COLOR NOTES

Each mode can have a geometrical representation, which can give information about its transformation relative to the others. This arises by taking the structure of a dodecaphonic scale, represented by a 12-segmented circle, in which each pitch is related to one of the points that segment the circle, where the base point is at the top of the figure.

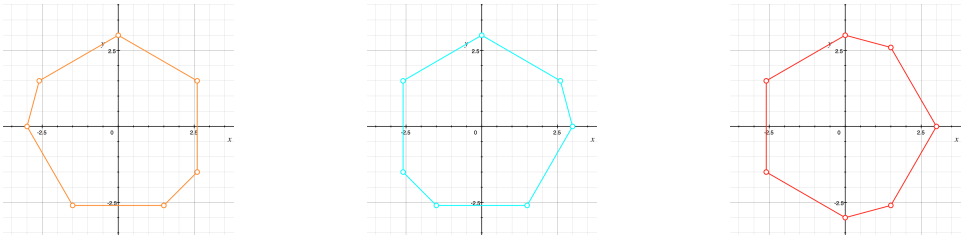
Then, the points of the circle will have some of them connected by the structure of that mode, as a projection of the mode into the dodecaphonic scale (see Figure 2).

Scale	Mode	Intervalic Set	$I(\overrightarrow{H})$
S_M	$\mathcal{I}$	[P1, M2, M3, P4, P5, M6, M7]	$H\triangle 13$
S_M	$\mathcal{D}$	[P1, M2, m3, P4, P5, M6, m7]	$H-13$
S_M	$\mathcal{P}$	[P1, m2, m3, P4, P5, m6, m7]	$H-11\flat 9\flat 13$
S_M	$\mathcal{L}$	[P1, M2, M3, a4, P5, M6, M7]	$H\triangle 13\sharp 11$
S_M	$\mathcal{X}$	[P1, M2, M3, P4, P5, M6, m7]	$H\triangle 13$
S_M	$\mathcal{A}$	[P1, M2, m3, P4, P5, m6, m7]	$H-11\flat 13$
S_M	$\mathcal{O}$	[P1, m2, m3, P4, d5, m6, m7]	$H^{\emptyset}11\flat 9\flat 13$
S_{mh}	$\mathcal{A}\ \triangle$	[P1, M2, m3, P4, P5, m6, M7]	$H-\triangle 13\flat 11$
S_{mh}	$\mathcal{OD}$	[P1, m2, m3, P4, d5, M6, m7]	$H^{\emptyset}13\flat 9$
S_{mh}	$\mathcal{I}+$	[P1, M2, M3, P4, a5, M6, M7]	$H+\triangle 13$
S_{mh}	$\mathcal{DL}$	[P1, M2, m3, a4, P5, M6, m7]	$H-13\sharp 11$
S_{mh}	$\mathcal{XP}\ \flat 13$	[P1, m2, M3, P4, P5, m6, m7]	$H11\flat 9\flat 13$
S_{mh}	$\mathcal{L}\ \sharp 9$	[P1, a2, M3, a4, P5, M6, M7]	$H\triangle 13\sharp 9\sharp 11$
S_{mh}	$\mathcal{O}\ \flat\flat 7\flat 11$	[P1, m2, m3, d4, d5, m6, d7]	$H^{o}7\flat 9\flat 11\flat 13$
S_{mm}	$\mathcal{D}\ \triangle$	[P1, M2, m3, P4, P5, M6, M7]	$H-\triangle 13$
S_{mm}	$\mathcal{DP}$	[P1, m2, m3, P4, P5, M6, m7]	$H-13\flat 9$
S_{mm}	$\mathcal{L}+$	[P1, M2, M3, a4, a5, M6, M7]	$H+\triangle 13\sharp 11$
S_{mm}	$\mathcal{LX}$	[P1, M2, M3, a4, P5, M6, m7]	$H13\sharp 11$
S_{mm}	$\mathcal{X}\ \flat 13$	[P1, M2, M3, P4, P5, m6, m7]	$H11\flat 13$
S_{mm}	$\mathcal{O}\ 2$	[P1, M2, m3, P4, d5, m6, m7]	$H^{\emptyset}11\flat 13$
S_{mm}	$\mathcal{O}\ \flat 11$	[P1, m2, m3, d4, d5, m6, m7]	$H^{\emptyset}7\flat 9\flat 11\flat 13$
S_{MH}	$\mathcal{I}\ \flat 6$	[P1, M2, M3, P4, P5, m6, M7]	$H\triangle 11\flat 13$
S_{MH}	$\mathcal{OD}\ 2$	[P1, M2, m3, P4, d5, M6, m7]	$H^{\emptyset}13$
S_{MH}	$\mathcal{P}\ \flat 11$	[P1, m2, m3, d4, P5, m6, m7]	$H-7\flat 9\flat 11\flat 13$
S_{MH}	$\mathcal{DL}\ \triangle$	[P1, M2, m3, a4, P5, M6, M7]	$H-\triangle 13\sharp 11$
S_{MH}	$\mathcal{XP}$	[P1, m2, M3, P4, P5, M6, m7]	$H13\flat 9$
S_{MH}	$\mathcal{L}\ +\sharp 9$	[P1, a2, M3, a4, a5, M6, M7]	$H+\triangle 13\sharp 9\sharp 11$
S_{MH}	$\mathcal{O}\ \flat\flat 7$	[P1, m2, m3, P4, d5, m6, d7]	$H^{o}11\flat 9\flat 13$

TABLE 2. Chord nomenclature for each mode of four scales



(1) Ionian (2) Dorian (3) Phrygian (4) Lydian



(5) Mixolydian (6) Aeolian (7) Locrian

FIGURE 2. Geometrization of major scale *homonym*-related modes

As might be expected, every mode can be seen as a geometric rotation, which was our description in Definition 2.3. But it is important to note that with our representation of a circular dodecaphonic scale, pitch $\hat{h}$ is the one at the top, which means that every mode would start on the same $\hat{h}$ pitch. On the same scale, this is not true because every mode is generated by every note on the scale, which means that every mode starts on different pitches ($\forall s_n \in S \Leftrightarrow \forall u \exists! \mu$). This is what is called a *relative*-related mode.

In the geometrical context, the representation of modes starting on the same pitch will be used to effectively observe some properties that arise from the superposition of modes. This is called a *homonym*-related mode (as in Figure 2). Both of these representations preserve the intervalic qualities of a mode. The only possible difference would arise by comparing different scales from different base notes, which is not our focus here.

Proposition 4.1. *Every relative-related mode rotates itself to transform into the next one exactly the distance to the base of the next mode. For example, in the transformation of Ionian to Dorian,*

$$\hat{1}(\mathcal{I}) \mapsto \hat{2}(\mathcal{I}) \Longleftrightarrow \hat{1}(\mathcal{I}) \mapsto \hat{1}(\mathcal{D}) \Longleftrightarrow \hat{2}(\mathcal{I}) \sim \hat{1}(\mathcal{D})$$

Which implies that $\forall(s_n \mapsto s_{n+1}) \Longleftrightarrow \mu_u \mapsto \mu_{u+1}, \qquad s_n \in \mu_u(S)$

Once we have a description of modal representation, we must define how color notes arise using harmonic nomenclature transformation and geometric superimposition.

Definition 4.2. *A color note γ of a mode is defined as an interval in the form q, s which defines the nature of a mode. This arises as the difference between modes of the same category, which have a generator mode $\delta(\mu^{\pm})$ for each one, and are described as a set of intervals*

- $I^{+}(\overrightarrow{H})$ *for major modes μ^{+} (modes in which $\hat{3} = M3$) which generator mode is $\delta(\mu^{+}) = \mu_0(S_M) = \mathcal{I} = H \triangle 13$ and*
- $I^{-}(\overrightarrow{H})$ *for minor modes μ^{-} (modes in which $\hat{3} = m3$) which generator mode is $\delta(\mu^{-}) = \mu_5(S_M) = \mathcal{A} = H - 11\flat 13$.*

Then, the color note for each category μ^{+} and μ^{-} is, respectively

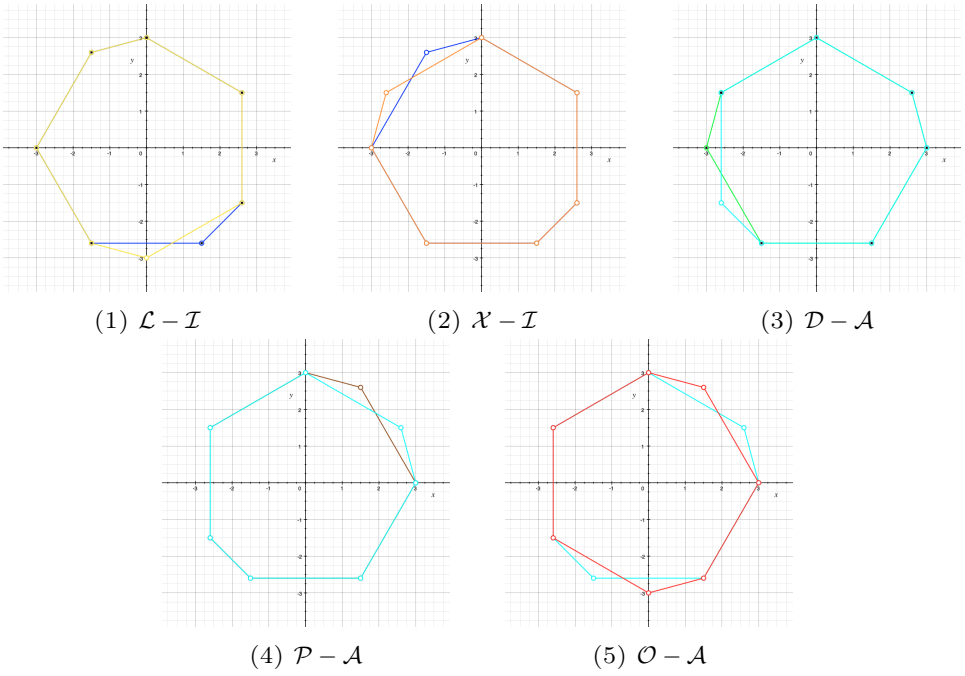
$$\begin{cases} \gamma(\mu^{+}) := \exists s_n \neq I^{+}(\overrightarrow{H}), & s_n \in \mu^{+} \\ \gamma(\mu^{-}) := \exists s_n \neq I^{-}(\overrightarrow{H}), & s_n \in \mu^{-} \end{cases}$$

Therefore, each category is $\mu^{+} = \{\mathcal{I}, \mathcal{L}, \mathcal{X}\}$, and $I\mu^{-} = \{\mathcal{A}, \mathcal{D}, \mathcal{P}, \mathcal{O}\}$.

We can calculate and geometrically represent these color notes as follows, by subtracting intervalic sets in their modulus-12 values for given *homonym*-related modes, therefore with the same h base note, where $-1 = \flat$ and $1 = \sharp$ for each $s_n(\mu^{\pm}) - s_n(I^{\pm}(\overrightarrow{H}))$.

These results will be shown below with their respective index related to their figure of geometrical representation based on superimposed *homonym*-related modes:

(1) $\mathcal{L} - I^{+}(\overrightarrow{H}) = \mathcal{L} - \mathcal{I} = [0,0,0,1,0,0,0] \implies \gamma(\mathcal{L}) = s_3 \simeq \sharp\hat{4} \simeq a4$
(2) $\mathcal{X} - I^{+}(\overrightarrow{H}) = \mathcal{X} - \mathcal{I} = [0,0,0,0,0,0,-1] \implies \gamma(\mathcal{X}) = s_6 \simeq \flat\hat{7} \simeq m7$
(3) $\mathcal{D} - I^{-}(\overrightarrow{H}) = \mathcal{D} - \mathcal{A} = [0,0,0,0,0,1,0] \implies \gamma(\mathcal{D}) = s_5 \simeq \natural\hat{6} \simeq M6$
(4) $\mathcal{P} - I^{-}(\overrightarrow{H}) = \mathcal{P} - \mathcal{A} = [0,-1,0,0,0,0,0] \implies \gamma(\mathcal{P}) = s_1 \simeq \flat\hat{2} \simeq m2$
(5) $\mathcal{O} - I^{-}(\overrightarrow{H}) = \mathcal{O} - \mathcal{A} = [0,-1,0,0,-1,0,0] \implies \gamma(\mathcal{O}) = \{s_1, s_4\} \sim \{\flat\hat{2}, \flat\hat{5}\} \simeq \{m2, d5\}$



(1) $\mathcal{L} - \mathcal{I}$ (2) $\mathcal{X} - \mathcal{I}$ (3) $\mathcal{D} - \mathcal{A}$
(4) $\mathcal{P} - \mathcal{A}$ (5) $\mathcal{O} - \mathcal{A}$

FIGURE 3. Geometrization of superimposed *homonym*-related modes

Lemma 4.3. *Locrian mode* ($\mathcal{O}$) *has two color notes* γ*, but as* $\mathcal{P}$ *and* $\mathcal{O}$ *are both from the minor category and both of them share the* $\flat\hat{2}$*, the only one taken as deterministic for* $\mathcal{O}$ *is the* $\flat\hat{5}$*.*

This way, color notes can be described as shown in Table 3 (these are the only color notes that are considered in the musical theory spectrum). Therefore, a determined mode will always be formed with the triad of that particular mode and the color note. This means that the quality in the minimal form of a mode can be reduced to 4 notes and even 3 in the particular case of $\mathcal{O}$ because its γ alters part of its triad. This is what we are going to call "*primary quality*" of a mode, described by its respective harmonic structure nomenclature.

Definition 4.4. *The primary quality of a mode* $Q(\mu)$ *is defined as the set*

$$Q(\mu) := \{\hat{1}, \hat{3}, \hat{5}\} \cup \gamma(\mu),$$

where $\forall Q(\mu) \setminus \hat{5} \iff \gamma(\mu) = \hat{5}$.

Proposition 4.5. *Though generator modes* $\delta(\mu^{\pm})$ *seem not to have a particular* γ*, and this statement is true for Definition 4.2, primary quality can also arise for these generator modes, which is*

$$Q(\mu(\delta(\mu^{\pm}))) = \{\hat{1}, \hat{3}, \hat{5}\} \cup \gamma^{-1}(\mu \neq \mu(\delta(\mu^{\pm}))),$$

where γ^{-1} *is the implicit form of any* γ*. For example,* $\gamma(\mathcal{X}) = m7 \implies \gamma^{-1}(\mathcal{X}) = M7$*,* $\gamma(\mathcal{P}) = m2 \implies \gamma^{-1}(\mathcal{P}) = M2$*, and so on.*

Proof. Any $\delta(\mu^{\pm})$ can have its primary quality if and only if there is no possible way to assume that it is a different mode, and as every γ describes the nature of each mode, if every $\gamma \in I^{\pm}(\overrightarrow{H})$, then the mode is necessary $\delta(\mu^{\pm})$. □

In Table 3, all color notes and primary qualities arising from Definition 4.4 and Proposition 4.5 will be shown.

Category	Mode	Symbol	$\gamma(\mu)$	Primary Quality
μ^+	Lydian	$\mathcal{L}$	a4	$Hadd4$
μ^+	Mixolydian	$\mathcal{X}$	m7	$H7$
μ^+	Ionian	$\mathcal{I}$	$\{P4, M7\}$	$H\triangle 7add4$
μ^-	Dorian	$\mathcal{D}$	M6	$H-6$
μ^-	Phrygian	$\mathcal{P}$	m2	$H-add\flat 2$
μ^-	Locrian	$\mathcal{O}$	d5	H^o
μ^-	Aeolian	$\mathcal{A}$	$\{M2, m6\}$	$H-add2, \flat 6$

TABLE 3. Color notes and primary quality for each mode

Primary quality allows us to make nomenclature of altered modes and combine them; for example, $\mathcal{LX}$ is a lydian structure with the color note of $\mathcal{X}$, which will be used to name $\mu_3(S_{mm}) \implies \overrightarrow{H}(\mathcal{LX}) = H13\sharp 11$. This way, we can describe a way to establish quality in a harmonic structure, which is through the *primary quality* of a given modal color.

This perspective is crucial to establish a structure and categorization of harmonic structures, which can lead us to interesting viewpoints applying transformational theory, also known as neo-Riemannian theory [10, 2, 5], in a more synthetic, abstract, and expanded way.

5. MODE TRANSFORMATION AND BRIGHTNESS COEFFICIENT

The transformation of a mode implies changes in its structural conformation. This can help to establish a structure of connections between them and, in this way, the relationship between modes of different scales and within a single scale can be seen.

A mode can transform into another one through one or more operators that change an element $s_n \in \mu$.

Definition 5.1. *An operator is defined as the transformation associated with an element of the mode which has 1) different magnitudes, and 2) can be positive or negative:*

$$\Omega := \pm\omega(s_n) \qquad \forall s_n \in \mu$$

where ω is the magnitude of the operator.

For example, we can have the transformations $\mathcal{X} \xrightarrow{-1(s_2)} \mathcal{D}$, $\mathcal{L}+ \xrightarrow{-1(s_3),-1(s_4)} \mathcal{I}$, $\mathcal{O} \xrightarrow{+1(s_1)} \mathcal{O}2$, $\mathcal{A}\triangle \xrightarrow{+1(s_2),+1(s_3),+1(s_5)} \mathcal{L}$, and so on. This will be used to relate modes with their transformational and their brightness coefficient relation.

Any structure can have a brightness coefficient given by the sum of its intervals relative to its base. This can give us a way to establish a hierarchical structure of brightness coefficients that affect the total energy of a musical passage. With these coefficients, we can also find superpositions across modes at different scales.

Definition 5.2. *The brightness coefficient is defined as the sum of the intervals of a structure from its base.*

$$B(S) = \sum_{k=0}^{n-1} s_k, \qquad S = \{s_0, s_1 \dots s_n\}$$

Lemma 5.3. *Every brightness coefficient will always be different for each mode within a single diatonic scale D:*

$$B(\mu_u(D)) \neq B(\mu_v(D)) \qquad \forall(\mu_u, \mu_v) \in D$$

The proof is a direct consequence of Proposition 2.5.

The brightness coefficient is a way to describe how we perceive a harmonic structure in terms of brightness or opacity. I found through practice that chords which are perceived as "dark" tend to have a broader separation for each component from the base of the structure. Brightness can also be an element that builds up what in music is known as "*energy*", which is the quality of a section, chord, or musical object in general to be perceived with more "*impact*", "*density*", "*size*", to relate some adjectives. The values of brightness coefficients in the modes of the four different scales will be shown in Table 4.

Therefore, we can establish a transformational-brightness hierarchical structure with the relations of modes $\Omega : \pm1(s_n)$ that are only one step of distance from each other with only one operator. This is what, in neo-Riemannian theory, is called a $P_{1,0}$ parsimonious transformation [2].

These relationships allow us to move not only in a transformational structural way, but also taking into account the brightness coefficient. Applying these structural concepts to music, energy can increase and decrease, transforming harmonic entities into others in a parsimonious way, or in other words, without losing the sense of stability.

In Figure 4, these transformational structures will be shown as a bi-dimensional representation, where each one of the vertical levels has different values of brightness coefficients which are separated by $\Omega : \pm1(s_n)$. Some values of brightness

Scale	Mode	s_0	s_1	s_2	s_3	s_4	s_5	s_6	Σ
S_{mh}	$\mathcal{I}$	0	2	4	5	7	9	11	38
S_M	$\mathcal{D}$	0	2	3	5	7	9	10	36
S_M	$\mathcal{P}$	0	1	3	5	7	8	10	34
S_M	$\mathcal{L}$	0	2	4	6	7	9	11	39
S_M	$\mathcal{X}$	0	2	4	5	7	9	10	37
S_M	$\mathcal{A}$	0	2	3	5	7	8	10	35
S_M	$\mathcal{O}$	0	1	3	5	6	8	10	33
S_{mh}	$\mathcal{A}\ \triangle$	0	2	3	5	7	8	11	36
S_{mh}	$\mathcal{OD}$	0	1	3	5	6	9	10	34
S_{mh}	$\mathcal{I}\ +$	0	2	4	5	8	9	11	39
S_{mh}	$\mathcal{DL}$	0	2	3	6	7	9	10	37
S_{mh}	$\mathcal{XP}\ \flat 13$	0	1	4	5	7	8	10	35
S_{mh}	$\mathcal{L}\ \sharp 9$	0	3	4	6	7	9	11	40
S_{mh}	$\mathcal{O}\ \flat\flat 7\flat 11$	0	1	3	4	6	8	9	31
S_{mm}	$\mathcal{D}\ \triangle$	0	2	3	5	7	9	11	37
S_{mm}	$\mathcal{DP}$	0	1	3	5	7	9	10	35
S_{mm}	$\mathcal{L}\ +$	0	2	4	6	8	9	11	40
S_{mm}	$\mathcal{LX}$	0	2	4	6	7	9	10	38
S_{mm}	$\mathcal{X}\ \flat 13$	0	2	4	5	7	8	10	36
S_{mm}	$\mathcal{O}\ 2$	0	2	3	5	6	8	10	34
S_{mm}	$\mathcal{O}\ \flat 11$	0	1	3	4	6	8	10	32
S_{MH}	$\mathcal{I}\ \flat 6$	0	2	4	5	7	8	11	37
S_{MH}	$\mathcal{OD}\ 2$	0	2	3	5	6	9	10	35
S_{MH}	$\mathcal{P}\ \flat 11$	0	1	3	4	7	8	10	33
S_{MH}	$\mathcal{DL}\ \triangle$	0	2	3	6	7	9	11	38
S_{MH}	$\mathcal{XP}$	0	1	4	5	7	9	10	36
S_{MH}	$\mathcal{L}\ +\sharp 9$	0	3	4	6	8	9	11	41
S_{MH}	$\mathcal{O}\ \flat\flat 7$	0	1	3	5	6	8	9	32

TABLE 4. Brightness coefficient in modes of four different scales

coefficients are, as seen before, equal across different scales. This is what we call a "*brightness-coefficient superposition*". These structures will also be shown in Figure 5, where the superposition of brightness coefficients is clearer.

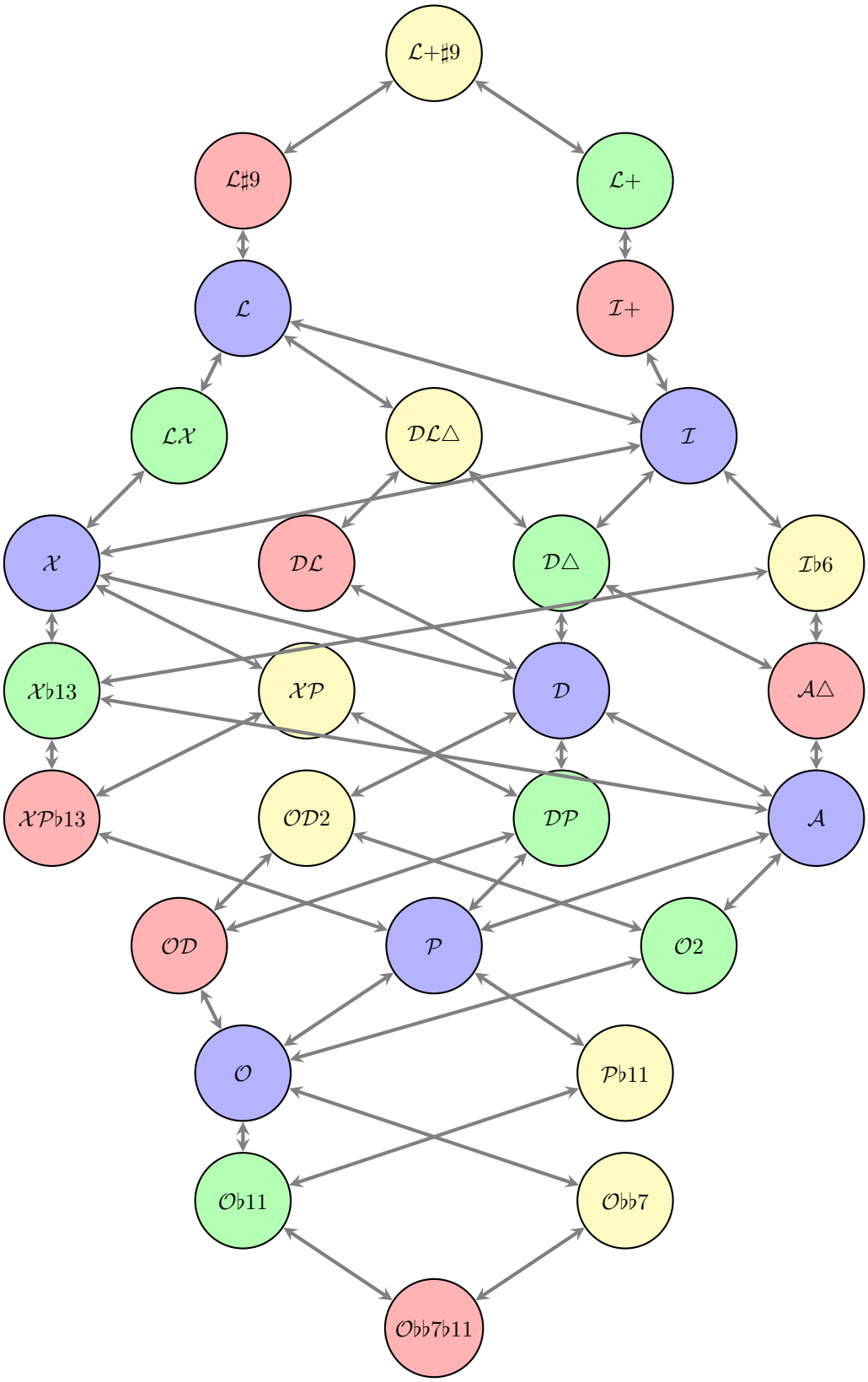
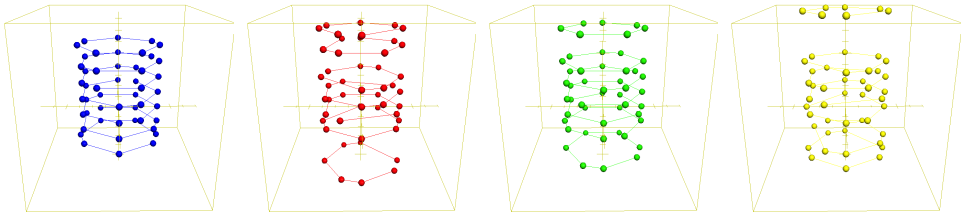
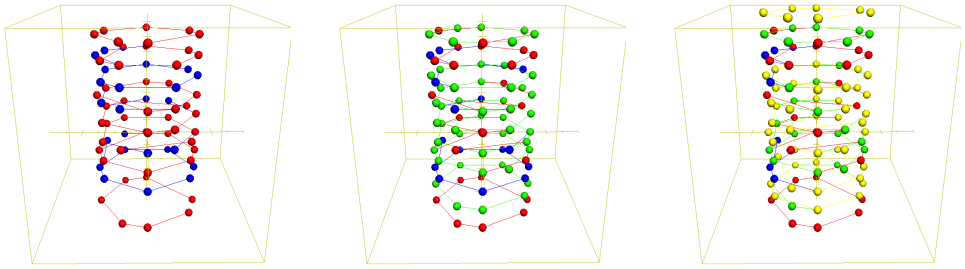


FIGURE 4. Transformational structure of brightness-parsimony related modes



(1) D_M Modes (2) D_{mh} Modes (3) D_{mm} Modes (4) D_{MH} Modes



(5) D_M and D_{mh} Superposition (6) D_M, D_{mh} and D_{mm} Superposition (7) D_M, D_{mh}, D_{mm} and D_{MH} Superposition

FIGURE 5. Superposition of brightness coefficients in different modes

6. SUPERPOSITION AND SYMMETRY OF MODES

Following the idea of superposition and brightness-coefficient differentiation between modes, we can also find an interesting relationship with maximally even structures.

Definition 6.1. *[1] A scale is maximally even if the spectrum of any generic interval consists either of a single integer or two consecutive integers. It is totally even if, additionally, $gcd(c, d) = d$, where $gcd(c, d)$ denotes the greatest common divisor of c and d.*

This way, the superposition of modes of a maximally even scale (D_M) that is totally even, and of D_{mm}, a maximally even scale, shows an interesting phenomenon regarding brightness coefficient ranges.

Proposition 6.2. *Two modes $\mu_u, \mu_{u'} \in D$, where D is a totally even scale which sum to twice the average of the total brightness coefficient of D, are always complementary opposites at the brightness coefficient ordering.*

Proof. For any totally even scale, every mode can transform into another with a $\Omega : \pm 1(s_n)$ transformation:

$$\mu_u \xrightarrow{\pm 1(s_n)} \mu_{u'}, \qquad \mu_u, \mu_{u'} \in D_M, \qquad s_n \in \mu_u,$$

therefore, the total sum of brightness coefficients can be

$$\sum_{k=0}^{n-1} B(\mu_k) = n \cdot \min B(\mu_k) + \sum_{k=0}^{n-1} k, \qquad n = |D| = 7,$$

which implies that

$$\min(B(\mu)) + \max(B(\mu)) = 2\min(B(\mu)) + n - 1, \qquad n = |D| = 7.$$

Additionally, we have that

$$\mu_u \xrightarrow{\pm 1(s_n)} \mu_{u'} \implies B(\mu) \to B(\mu) \pm 1,$$

which implies that

$$\min(B(\mu)) + k + \max(B(\mu)) - k = \min(B(\mu)) + \max(B(\mu)).$$

Where $k = 0 \dots n-1, \qquad n = |D| = 7.$

Therefore, for any complementary opposite modes, the result is always the same. These results can also be seen in Table 4, where it is also compiled for maximally even scales, with the difference that k is a set of intervals, which preserves symmetry, which implies that $k - k = 0$ also occurs. □

This leads us to one of the most important phenomena in this topic. It is very common to hear that some harmonic structures are "chromatically opposite" to each other, or that some modes are inverse of others. Also, it is somewhat described in a geometrical way, but the missing piece is the brightness coefficient. This way, we can have a very interesting phenomenon regarding 1) orientation, 2) geometrical complement, and 3) brightness neutralization.

Theorem 6.3 (Theorem of harmonic complementary color)**.** *Two superimposed modes (union) on the same maximally even or totally even scale whose average brightness coefficient equals the average brightness coefficient of the whole scale form a reversible structure with at least 2 symmetries from the base note; this is what we call a chromatic complementary relation ρ; we say two structures are ρ-related if:*

$$\frac{B(\mu_u) + B(\mu_{u'})}{2} = \frac{\sum_{k=0}^{n-1} B(\mu_k)}{n-1} \implies \mu_u \cup \mu_{u'} = -(\mu_u \cup \mu_{u'}) \implies \mu_u = -\mu_{u'},$$

where $\mu_u, \mu_{u'} \in D$.

Proof. For two modes that comply with Proposition 6.2, we can see that, for example:

- $\mathcal{O} \cup \mathcal{L} = \{0,1,3,5,6,8,10\} \cup \{0,2,4,6,7,9,11\} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$
- $\mathcal{P} \cup \mathcal{I} = \{0,1,3,5,7,8,10\} \cup \{0,2,4,5,7,9,11\} = \{0,1,2,3,4,5,7,8,9,10,11\}$
- $\mathcal{DP} \cup \mathcal{D}\triangle = \{0,1,3,5,7,9,10\} \cup \{0,2,3,5,7,9,11\} = \{0,1,2,3,5,7,9,10,11\}$
- $\mathcal{O}2 \cup \mathcal{LX} = \{0,2,3,5,6,8,10\} \cup \{0,2,4,6,7,9,10\} = \{0,1,2,3,4,5,6,7,8,9,10\}$

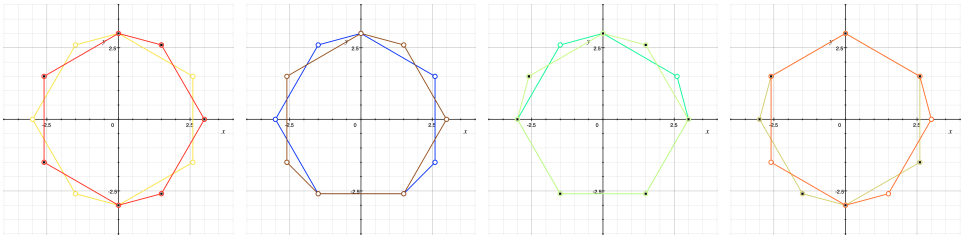
Where we can also see that as every element is in $mod 12$:

- $\mathcal{O} \cup (-\mathcal{L}) = \{0,1,3,5,6,8,10\} \cup -\{0,2,4,6,7,9,11\} = \{0,1,3,5,6,8,10\} \cup \{0,10,8,6,5,3,1\} = \{0,1,3,5,6,8,10\} = \mathcal{O}$

- $\mathcal{P} \cup (-\mathcal{I}) = \{0,1,3,5,7,8,10\} \cup -\{0,2,4,5,7,9,11\} = \{0,1,3,5,7,8,10\} \cup \{0,10,8,7,5,3,1\} = \{0,1,3,5,7,8,10\} = \mathcal{P}$
- $\mathcal{DP} \cup (-\mathcal{D})\triangle = \{0,1,3,5,7,9,10\} \cup -\{0,2,3,5,7,9,11\} = \{0,1,3,5,7,9,10\} \cup \{0,10,9,7,5,3,1\} = \{0,1,3,5,7,9,10\} = \mathcal{DP}$
- $\mathcal{O}2 \cup (-\mathcal{LX}) = \{0,2,3,5,6,8,10\} \cup -\{0,2,4,6,7,9,10\} = \{0,2,3,5,6,8,10\} \cup \{0,10,8,6,5,3,2\} = \{0,2,3,5,6,8,10\} = \mathcal{O}2$

This way, we can infer by mathematical induction that it works for pairs of modes that comply with Proposition 6.2 for every maximally and totally even scale. □

The result of a structure whose orientation and brightness are neutralized with the union of the other is what can be associated with what, in color theory, is called the complementary color: a superimposition of two colors that neutralize each other [15]. These results can also be seen in Figure 6.



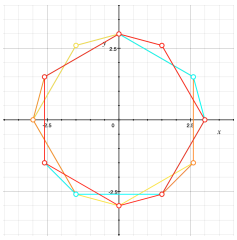
(1) Locrian and Lydian Superposition (2) Phrygian and Ionian Superposition (3) Dorian ♭9 and Lydian △ Superposition (4) Locrian2 and Lydian ♭7 Superposition

FIGURE 6. Superposition of Complementary Opposite Modes

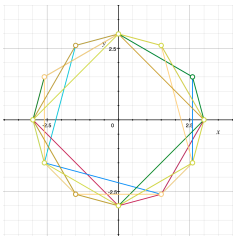
Theorem 6.4 (Theorem of unaltered mediants)**.** *The structure of superimposed homonym-related modes of the 4 scales presented shows a structure where the intervalic fluctuation for non-mediant degrees with respect to the base* $(\hat{2}, \hat{4}, \hat{5}, \hat{7})$ *varies between* $i, i^\star$ *(non-altered, altered) intervals, while mediant degrees* $(\hat{3}, \hat{6})$ *are always* i *(non-altered) intervals. The proof is a direct consequence of the geometrical representation of Figure 7, where it can be seen that every mediant interval is never at the same position as a non-altered non-mediant interval:*

$$\cup(M(D_M, D_{mh}, D_{mm}, D_{MH})) \implies \{\hat{3}, \hat{6}\} \iff \neg i^\star$$

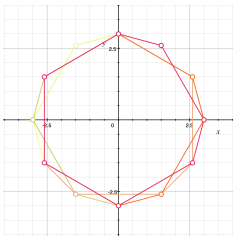
where M *is the set of modes* μ *of each scale.*



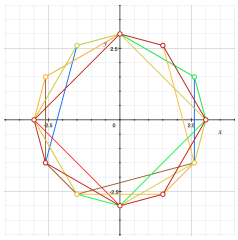
(1) D_M superimposed modes



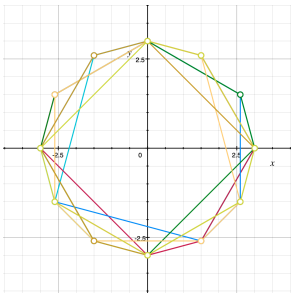
(2) D_{mh} superimposed modes



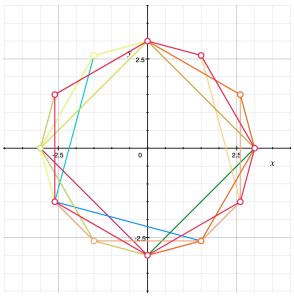
(3) D_{mm} superimposed modes



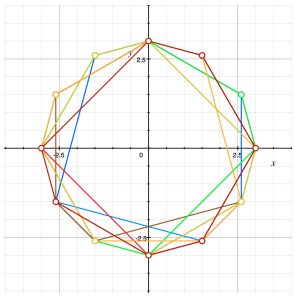
(4) D_{MH} superimposed modes



(5) D_M and D_{mh} Superposition



(6) D_M, D_{mh} and D_{mm} Superposition



(7) D_M, D_{mh}, D_{mm} and D_{MH} Superposition

FIGURE 7. Superposition of different modes structure

Superposition of the full spectrum of modes can show how they relate to each other and how they differ; for example, it is clear how modes from harmonic major and minor scales have less symmetrical and wider connections between them, in contrast to how maximally and totally even scales look in their geometrical distribution.

7. Conclusions and results

Geometrical representation of modes enriched with the comparison of their respective brightness coefficients and their transformational processes gives us some results about:

(1) The primary quality of a mode $Q(\mu)$ that is built with the triad and color note, which was defined as the set

$$Q(\mu) := \{\hat{1}, \hat{3}, \hat{5}\} \cup \gamma(\mu), \qquad \forall Q(\mu) \setminus \hat{5} \iff \gamma(\mu) = \hat{5}.$$

(2) We can also use *brightness equivalence states* to move from one scale to another, or transform the structural sensation maintaining the same brightness of it using an operator which has 1) different magnitudes, and 2) can be positive or negative:

$$\Omega := \pm\omega(s_n) \qquad \forall s_n \in \mu$$

where ω is the magnitude of the operator.

(3) How pairs of modes can complement each other to form a 2-sided symmetrical structure and neutralize their brightness and orientation within each other

$$\frac{B(\mu_u) + B(\mu_{u'})}{2} = \frac{\sum_{k=0}^{n-1} B(\mu_k)}{n-1} \implies \mu_u = -\mu_{u'},$$

where $\mu_u, \mu_{u'} \in D$.

(4) How some intervals remain unaltered with the full superposition of modes of the 4 scales, which opens the possibility of discovering the nature of mediant degrees as qualitative determinants:

$$\cup(M(D_M, D_{mh}, D_{mm}, D_{MH})) \implies \{\hat{3}, \hat{6}\} \iff \neg i^\star$$

where M is the set of modes μ of each scale.

Some of these results can determine a new way to understand the nature of harmonic structures, not only how they can transform into others, but also their qualities, which were described here as a formalization of brightness and color.

The nature of the primary quality of a mode is something that, as jazz musicians (mostly) or contemporary classical composers, is experienced through practical experience. With our results of Theorem 6.4, it can be inferred that the concept of primary quality can be generalized, not only to modes but also to the general concept of "harmonic structure", that is ¿when does a harmonic structure have a differentiable quality, and which properties are they?

I believe that these concepts will open a perspective for new study matter about harmony, and clarify some properties of what musicians and composers conceptualize about them in practice, in addition to observing the beauty of harmonic phenomena and the space which arises with every structure and transformation in it. Every table

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